Resolving Inconsistencies in Simple Temporal Problems

A Parameterized Approach

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Overview

- *Simple Temporal Problem* (STP) is an influential formalism for encoding and reasoning about temporal relations.
- STP constraints: $a \leq x_i - x_j \leq b$, where x_i, x_j represent points in time and a, b are rational or infinite values.
- STP consistency can be checked in polynomial time.
- But what if STP constraints are inconsistent?
- We study Almost STP: the problem of resolving few inconsistencies using tools from *parameterized complexity*.
- For two large classes of STP constraints (one-sided and equation constraints), we find fpt algorithms.
- We determine complexity of all classes of STP constraints.

Simple Temporal Problem (STP)

Introduced by Dechter, Meiri, and Pearl in 1989.

Objects: points in time $x_1, x_2, \ldots, x_n$.

Constraints: $a \leq x_i - x_j \leq b$, where $a, b \in \mathbb{Q} \cup \{-\infty, \infty\}$.

Examples of constraints:

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\begin{aligned}
1 \leq x_i - x_j &\leq 2, \\
-\infty \leq x_i - x_j &\leq -2 && \text{(one-sided)}, \\
1 \leq x_i - x_j &\leq \infty && \text{(one-sided)}, \\
1 \leq x_i - x_j &\leq 1 \quad \equiv x_i - x_j = 1 && \text{(equation)}.
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Simple Temporal Problem (STP)

Checking consistency requires polynomial time.

$$\begin{aligned}
-1 &\le d - a \le 4\\
2 &\le b - a \le 4\\
2 &\le c - b \le 3\\
1 &\le e - d \le 2\\
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Consistent if and only if contains no negative cycles.

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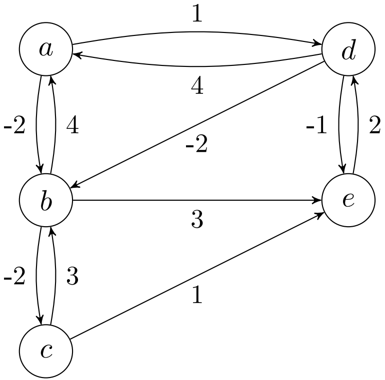

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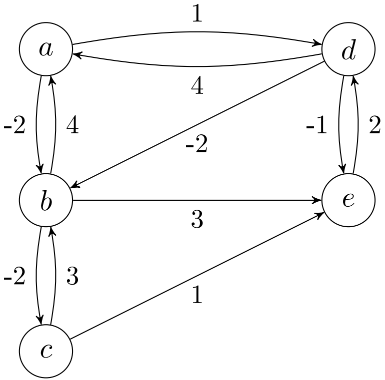

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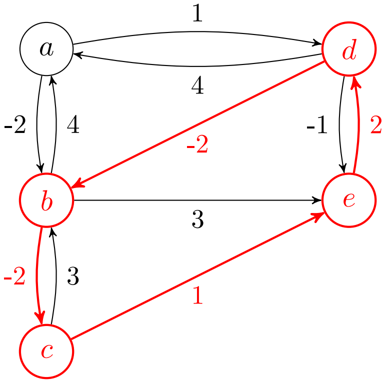

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Almost STP

- How to deal with with inconsistent instances?
- *Remove some constraints to achieve consistency.*
- Call this problem Almost STP.
- Almost STP is NP-hard.
- *Restrict the set of allowed constraints.*
- Almost STP is in P only when restricted to trivial constraints ($a \leq x_i - x_j \leq b$, where $a \leq 0 \leq b$) and NP-hard otherwise.
- *Assume that removing few constraints is enough.*
- Study complexity of Almost STP parameterized by k – number of constraints to be removed.

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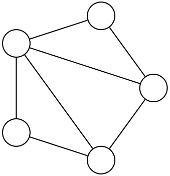
Parameterized Complexity

k-Vertex Cover

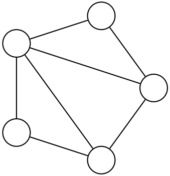


Cover all edges with k vertices.

Solvable in $f(k) \cdot \mathrm{poly}(n)$ time.

In FPT.

k-Independent Set



Find k non-adjacent vertices.

Solvable in $n^{O(k)}$ time.

W[1]-hard.

Conjecture

FPT $\neq$ W[1]

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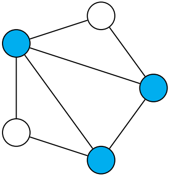

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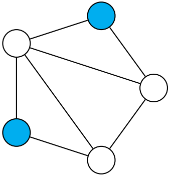

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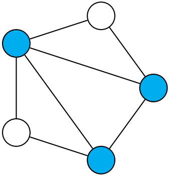

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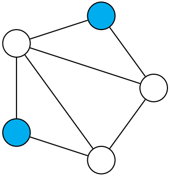

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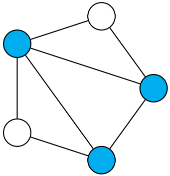

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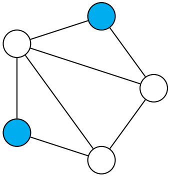

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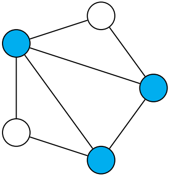

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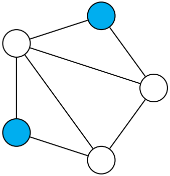

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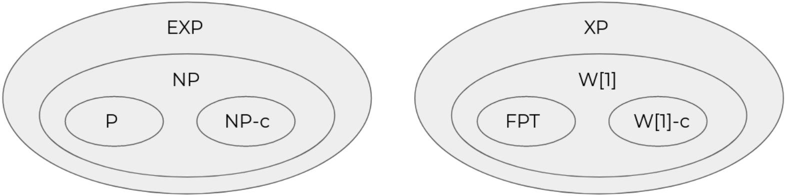

Back to ALMOST STP

- Let $\mathcal{S}$ contain $a \leq x_i - x_j \leq b$ for all $a, b \in \mathbb{Q} \cup \{-\infty, \infty\}$.
- For every subset $\mathcal{A}$ of $\mathcal{S}$, what is the parameterized complexity of ALMOST STP restricted to $\mathcal{A}$?
- Some subsets of $\mathcal{S}$:
  - Trivial constraints: $a \leq x_i - x_j \leq b$, where $a \leq 0 \leq b$.
  - One-sided constraints: $a \leq x_i - x_j$, where $a \geq 0$.
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- $1 \leq x_i - x_j \leq 2$ is not trivial, one-sided or equation.

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Classification Theorem

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ALMOST STP *restricted to* $\mathcal{A} \subseteq \mathcal{S}$ *is*

1. *in constant time if* $\mathcal{A}$ *only contains trivial constraints,*
2. *in FPT if* $\mathcal{A}$ *only contains one-sided constraints,*
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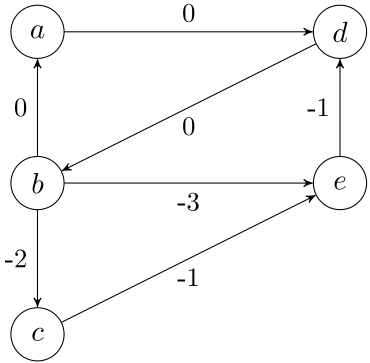
One-sided constraints

Examples: $0 \leq d - a$, $1 \leq d - e$, $2 \leq c - b$, ...



- At most one arc for every pair.
- Labels either zero or negative.
- Negative cycles are bad.
- Zero cycles are OK.
- All cycles with at least one negative arc are bad.
- **Goal:** find k arcs that intersect every cycle with a negative arc.
- In FPT by Chitnis et al.

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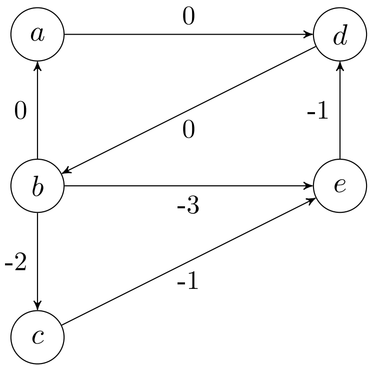

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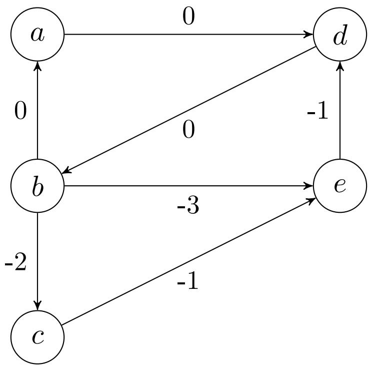

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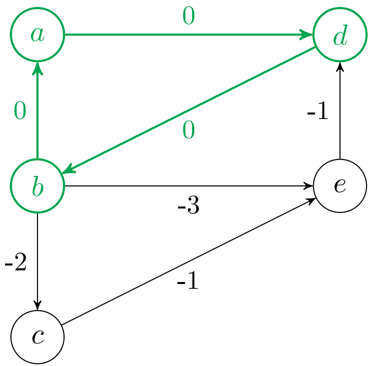

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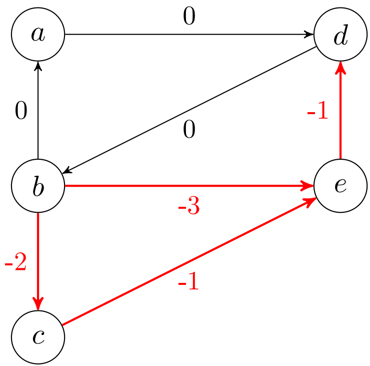

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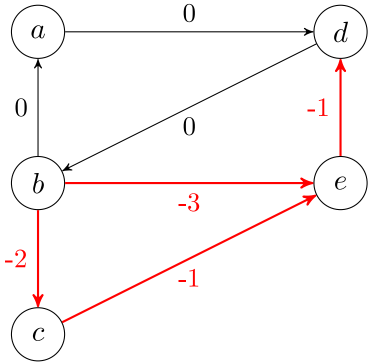

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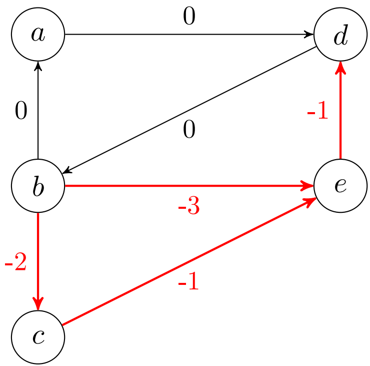

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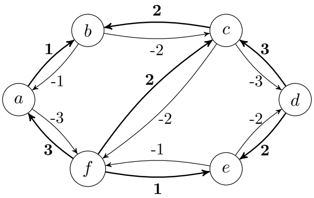

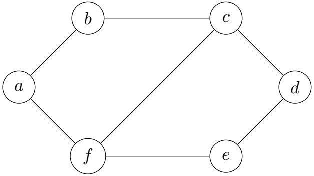

- $a - b = 1$: $a \xrightarrow{1} b$, $b \xrightarrow{-1} a$.
- Values propagate.
- Nonzero cycles are bad.
- **Goal:** find k arcs that intersect every nonzero cycle.
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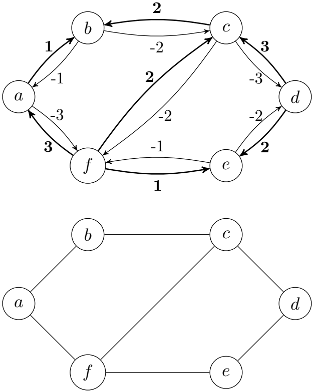

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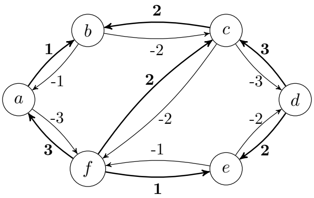

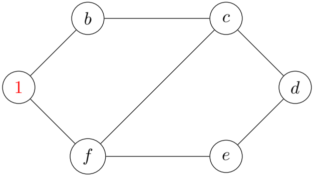

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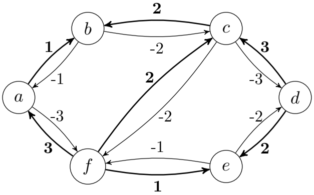

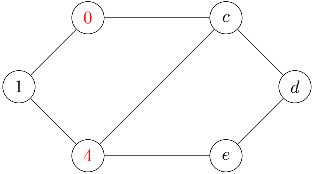

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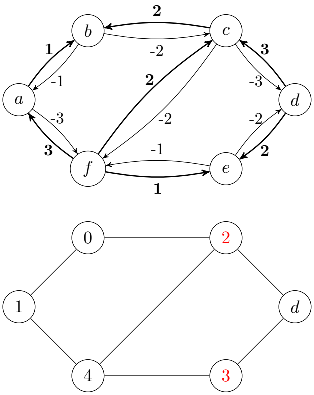

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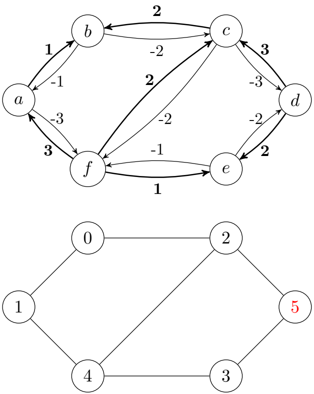

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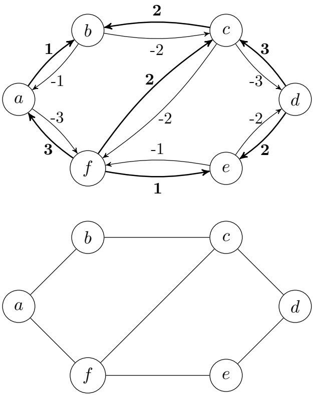

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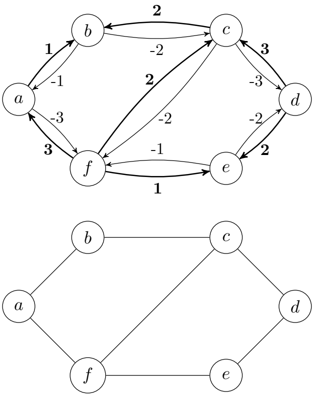

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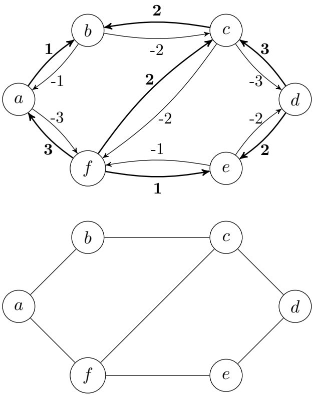

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Classification Theorem

Theorem

ALMOST STP *restricted to* $\mathcal{A} \subseteq \mathcal{S}$ *is*

1. *in constant time if* $\mathcal{A}$ *only contains trivial constraints,*
2. *in FPT if* $\mathcal{A}$ *only contains one-sided constraints,*
3. *in FPT if* $\mathcal{A}$ *only contains equation constraints, and*
4. *W[1]-hard otherwise.*

W[1]-hard Cases (1/3)

Theorem (Göke et al.)

If $\mathcal{A}$ contains $x_i - x_j \leq 1$ and $x_i - x_j \geq 1$, then AlmostSTP *restricted to $\mathcal{A}$ is W[1]-hard.*

- $x_i - x_j \leq 2$ and $x_i - x_j \geq 2$ imply W[1]-hardness.
- What about $x_i - x_j \leq 2$ and $x_i - x_j \geq 3$?
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If $\mathcal{A}$ contains $x_i - x_j \leq a$ and $x_i - x_j \geq b$ for any $a, b \in \mathbb{Q}_{>0}$, then ALMOSTSTP *restricted to $\mathcal{A}$ is W[1]-hard.*

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W[1]-hard Cases (3/3)

Lemma

If $\mathcal{A}$ *contains*
(a) $x_i - x_j \le a$ *and* $x_i - x_j \ge b$ *for any* $a, b \in \mathbb{Q}_{>0}$, *or*
(b) $a \le x_i - x_j \le b$ *for some* $0 < a < b < \infty$,
then AlmostSTP *restricted to* $\mathcal{A}$ *is W[1]-hard.*

Finally, we prove that if $\mathcal{A}$ is not trivial, one-sided, or equation, then it either implements two constraints from (a) or the constraint from (b).

W[1]-hard Cases (3/3)

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Finally, we prove that if $\mathcal{A}$ is not trivial, one-sided, or equation, then it either implements two constraints from (a) or the constraint from (b).

Questions for Future

- What if we allow unary constraints, e.g. $1 \leq x_i \leq 3$?
- What if we allow strict constraints, e.g. $1 < x_i - x_j \leq 2$?
- For which other problems X is ALMOST X interesting?
- ALMOST STP assumes that the *additive* error is small. What about the *multiplicative* error? Can we check if $(1 - \epsilon)$ fraction of STP constraints are consistent? This question is asking about *robust approximation*.

Thank you!